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Short Communication

Semi-inverse method for axially functionally graded beams with an anti-symmetric vibration mode

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Abstract

In this paper, we use the semi-inverse method to find the solutions to the dynamic equation of inhomogeneous, functionally graded simply supported beams. For the anti-symmetric mode and the material density (or the Young's modulus) that are both polynomial functions and and have been pre-specified, we find the Young's modulus (or the density) in polynomial functions and determine the closed from expression for the natural frequency.

1. Introduction

The vibration of structures made of functionally graded materials (FGM) attracted much attention recently. FGM are materials that exhibit continuous smooth variation of the elastic modulus in the thickness direction in order to be able to design structures with desired characteristics. The papers by Loy et al. [1], Vel and Batra [2], Yang and Shen [3,4], Cheng and Batra [5], Chen and Ding [6] deal with various vibration problems of the structures made of the FGM. The functional grading in the axial direction, i.e. the variation of the elastic modulus along the axis, was studied by Candan and Elishakoff [7], Elishakoff and Candan [8], and Guede and

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Elishakoff [9]. The problematics described in these papers are intimately connected with the studies by Wang et al. [10], and Wang and Wang [11,12].

Neuringer and Elishakoll [13,14] considered the case where the axially graded structure has a prescribed polynomial second mode. In this paper, we investigate this topic in more detail.

2. Problem description

The dynamic governing equation of an inhomogeneous beam reads

$$\frac{d^2}{d\xi^2}\left[D(\xi)\,\frac{d^2 w(\xi)}{d\xi^2}\right] - kL^4 R(\xi) w(\xi) = 0, \tag{1}$$

where $D = EI$ is the flexural stiffness, E the Young's modulus, I the moment of inertia of the cross section, $R = \rho A$ the inertial coefficient, ρ the density, A the area of the cross-section, w the displacement, $k = \omega^2$, ω the natural frequency, $\xi = x/L$ a non-dimensional axial coordinate, and L is the length.

Since a large number of functionally graded materials is emerging, the beam of variable parameters is expected to be utilized much more widely in engineering than at present. Thus, the dynamic problem of an inhomogeneous beam becomes of prime importance. In this paper, we investigate the case when the inertial coefficient $R(\xi)$ is given, and find the flexural stiffness $D(\xi)$ and the corresponding natural frequency ω, when the second mode (postulated as the static displacement of homogeneous beam under the action of anti-symmetric linear load) is known. The result indicated that two kinds of inverse problems formulated here can be solved in the closed form if some condition is satisfied.

3. Basic equations

It is assumed that the cross-sectional area A is constant, whereas $R(\xi)$, $D(\xi)$ and $w(\xi)$ are polynomial functions given by

$$R(\xi) = \sum_{i=0}^{m} a_i \xi^i, \quad D(\xi) = \sum_{i=0}^{n} b_i \xi^i, \quad w(\xi) = \sum_{i=0}^{p} w_i \xi^i, \tag{2}$$

where m, n, and p are the degree of the polynomials for $R(\xi)$, $D(\xi)$ and $w(\xi)$, respectively.

As the involved functions are assumed to be polynomial, the degrees of the above polynomial functions must be linked, namely

$$n + (p-2) - 2 = m + p \tag{3}$$

or simply, $n - m = 4$.

Thus, $D(\xi)$ can be written as

$$D(\xi) = \sum_{i=0}^{m+4} b_i \xi^i. \tag{4}$$

We observe that Eq. (3) is not dependent on the degree p of the displacement polynomial $w(\xi)$; therefore, any polynomial function for the displacement may be used in Eq. (1) if, obviously, it also satisfies the boundary conditions. The following boundary conditions in addition to the conditions defining the anti-symmetric property of the mode of a simply supported beam must be satisfied:

$$w(0) = 0, \quad D(0)w''(0) = 0, \quad w(1) = 0, \quad D(1)w''(1) = 0, \tag{5}$$

and

$$w\left(\tfrac{1}{2}\right) = 0. \tag{6}$$

Eq. (5) allows for Young's modulus to assume a zero value at $\xi = 0$ or $\zeta = 1$. This, however, makes no physical sense. Hence, Eq. (5) is replaced with the following requirements:

$$w(0) = 0, \quad w''(0) = 0, \quad w(1) = 0, \quad w''(1) = 0. \tag{7}$$

Satisfaction of Eqs. (6) and (7) requires the order of the displacement polynomial to be at least five. Assuming that $w(\xi)$ is a fifth-order polynomial,

$$w(\xi) = \xi_0 + w_1\xi_1 + w_2\xi^2 + w_3\xi^3 + w_4\xi^4 + w_5\xi^5, \tag{8}$$

the satisfaction of the boundary conditions yields

$$w(\xi) = w_1(\xi - 10\xi^3 + 15\xi^4 - 6\xi^5), \tag{9}$$

where w_1 is an arbitrary coefficient. By substituting the expressions of $D(\xi), R(\xi), w(\xi)$ in Eq. (1), we obtain

$$\begin{aligned} & w_1 \sum_{i=2}^{m+4} i(i-1)b_i\xi^{i-2}60(-\xi + 3\xi^2 - 2\xi^3) \\ & \quad + w_1 \sum_{i=0}^{m+4} b_i\xi^i 360(1 - 2\xi) + w_1 2 \sum_{i=1}^{m+4} ib_i\xi^{i-1}60(-1 + 6\xi - 6\xi^2) \\ & \quad = w_1 kL^4 \sum_{i=0}^{m} a_i\xi^i(\xi - 10\xi^3 + 15\xi^4 - 6\xi^5). \end{aligned} \tag{10}$$

The latter expression can be rewritten as

$$\begin{aligned} & -60 \sum_{i=1}^{m+3} i(i+1)b_{i+1}\xi^i + 180 \sum_{i=2}^{m+4} i(i-1)b_i\xi^i - 120 \sum_{i=3}^{m+5} (i-1)(i-2)b_{i-1}\xi^i + 360 \sum_{i=0}^{m+4} b_i\xi^i \\ & \quad - 720 \sum_{i=1}^{m+5} b_{i-1}\xi^i - 120 \sum_{i=0}^{m+3} (i+1)b_{i+1}\xi^i + 720 \sum_{i=1}^{m+4} ib_i\xi^i - 720 \sum_{i=2}^{m+5} (i-1)b_{i-1}\xi^i \\ & \quad - kL^4 \sum_{i=1}^{m+1} a_{i-1}\xi^i + 10kL^4 \sum_{i=3}^{m+3} a_{i-3}\xi^i - 15kL^4 \sum_{i=4}^{m+4} a_{i-4}\xi^i + 6kL^4 \sum_{i=5}^{m+5} a_{i-5}\xi^i = 0. \end{aligned} \tag{11}$$

Eq. (11) has to be satisfied for any ξ. This requirement yields the following relations:

$$-120(3b_0 - b_1) = 0, \tag{12}$$

$$-kL^4a_0 - 60 \times 2 \times 3(2b_0 - 3b_1 + b_2) = 0, \tag{13}$$

$$-kL^4a_1 - 60 \times 3 \times 4(2b_1 - 3b_2 + b_3) = 0, \tag{14}$$

$$kL^4(10a_0 - a_2) - 60 \times 4 \times 5(2b_2 - 3b_3 + b_4) = 0, \tag{15}$$

$$kL^4(-15a_0 + 10a_1 - a_3) - 60 \times 5 \times 6(2b_3 - 3b_4 + b_5) = 0, \tag{16}$$

$$kL^4(6a_{i-5} - 15a_{i-4} + 10a_{i-3} - a_{i-1}) - 60(i+1)(i+2)(2b_{i-1} - 3b_i + b_{i+1}) = 0, \quad 5 \leqslant i \leqslant m+1; \tag{17}$$

$$kL^4(6a_{m-3} - 15a_{m-2} + 10a_{m-1}) - 60(m+3)(m+4)(2b_{m+1} - 3b_{m+2} + b_{m+3}) = 0, \tag{18}$$

$$kL^4(6a_{m-2} - 15a_{m-1} + 10a_m) - 60(m+4)(m+5)(2b_{m+2} - 3b_{m+3} + b_{m+4}) = 0, \tag{19}$$

$$kL^4(6a_{m-1} - 15a_m) - 60(m+5)(m+6)(2b_{m+3} - 3b_{m+4}) = 0, \tag{20}$$

$$6kL^4a_m - 60(m+6)(m+7)2b_{m+4} = 0. \tag{21}$$

Note that Eqs. (12)–(21) are valid only if $m \geqslant 4$. The cases which satisfy the inequality $m < 4$ will be discussed at the later stage. According to previous equations, and bearing in mind the expression of the displacement mode $w(\xi)$ specified in Eq. (9), we consider two separate problems: (1) material density coefficients a_i are specified, find coefficients b_i in the elastic modulus representation; (2) elastic modulus coefficients b_i are specified, determine the coefficients a_i so that a closed-form solution is obtainable.

4. Specified inertial coefficient function

Let us consider the case when the function $R(\xi)$ of the inertial coefficient is known. This implies that all coefficients a_i are given. It must be remarked that the coefficients a_i cannot be equal to each other, otherwise in Eq. (17) the coefficient in front of k would vanish. Eqs. (12)–(20) can be written as matrix equation:

$$60\mathbf{D} \cdot (\mathbf{C} \cdot \mathbf{B}) = kL^4\mathbf{E} \cdot \mathbf{A}, \tag{22}$$

where $\mathbf{D}$ is $m+5$ tri-diagonal square matrix,

$$\mathbf{D} = \mathrm{diag}\{1 \cdot 2, 2 \cdot 3, \ldots, i(i+1), \ldots, (m+5)(m+6)\}, \tag{23}$$

$$
\mathbf{C} = \begin{bmatrix}
3 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-2 & 3 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -2 & 3 & -1 & \cdots & 0 & 0 & 0 & 0 \\
 & \vdots & & \vdots & & \vdots & & \vdots & \\
0 & 0 & 0 & 0 & \cdots & -2 & 3 & -1 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -2 & 3 & -1 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -2 & 3
\end{bmatrix}. \tag{24}
$$

$\mathbf{C}$ is the Jacob matrix of $m+5$-order, whereas

$$
\mathbf{B} = (b_0, b_1, b_2, \ldots, b_{m+3}, b_{m+4})^{\mathrm{T}}, \quad \mathbf{A} = (a_0, a_1, a_2, \ldots, a_{m-1}, a_m)^{\mathrm{T}}. \tag{25}
$$

Namely, $\mathbf{B}$ is an $m+5$ dimensional column vector. $\mathbf{A}$ is an $m+1$ dimensional column vector.

Moreover, $\mathbf{E}$ is the matrix $m+5$ of the row, $m+1$ of the column, and

$$
\mathbf{E} = \begin{bmatrix}
0 & \cdots & & \cdots & & \cdots & & \cdots & 0 \\
1 & 0 & \cdots & & \cdots & & \cdots & & 0 \\
0 & 1 & 0 & \cdots & & \cdots & & \cdots & 0 \\
-10 & 0 & 1 & 0 & \cdots & & \cdots & & 0 \\
15 & -10 & 0 & 1 & 0 & \cdots & & \cdots & 0 \\
-6 & 15 & -10 & 0 & 1 & 0 & \cdots & & 0 \\
0 & -6 & 15 & -10 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & -6 & 15 & -10 & 0 & 1 & 0 \\
0 & & \cdots & 0 & -6 & 15 & -10 & 0 & 1 \\
0 & \cdots & & \cdots & 0 & -6 & 15 & -10 & 0 \\
0 & & \cdots & & \cdots & 0 & -6 & 15 & -10 \\
0 & \cdots & & \cdots & & \cdots & 0 & -6 & 15
\end{bmatrix}. \tag{26}
$$

It is easy to check that $\mathbf{C}$ is reversible as a sign-oscillating matrix:

$$
\mathbf{c}^{-1} = \frac{1}{2^{(m+5)+1} - 1} \{\alpha_{ij}\}_{(m+5)\times(m+5)}, \tag{27}
$$

where

$$
\alpha_{ij} = \begin{cases} (2^{m+5-j+1} - 1)(2^i - 1), & i \leqslant j, \\ (2^{m+5-i+1} - 1)(2^i - 2^{i-j}), & i > j. \end{cases} \tag{28}
$$

So, we can write $\mathbf{c}^{-1}$ as

$$\mathbf{c}^{-1}=\frac{1}{2^{n+1}-1}\begin{bmatrix} 2^n-1 & 2^{n-1}-1 & 2^{n-2}-1 & \cdots & \cdots & 15 & 7 & 3 & 1 \\ 2^n-2 & & & & & & & & 3 \\ 2^n-4 & \ddots & & & & & & & 5 \\ \vdots & & \ddots & & & \alpha_{ij}(i\leqslant j) & & & 7 \\ \vdots & & & \ddots & & & & & 15 \\ \vdots & & & & \ddots & & & & \vdots \\ \vdots & & \alpha_{ij}(i>j) & & & \ddots & & & \vdots \\ 2^n-2^{n-2} & & & & & & \ddots & & 2^{n-1}-1 \\ 2^{n-1} & \cdots & & & & \cdots & 2^n-4 & 2^n-2 & 2^n-1 \end{bmatrix}_{n\times n}. \tag{29}$$

Therefore, $\mathbf{B}$ has a closed-form unique solution:

$$b_0=b_1/3, \quad b_i=\frac{kL^4}{60}f_i(a_0,a_1,a_2,\ldots,a_{m-1},a_m,m) \quad (i=1,\ldots,m+4). \tag{30}$$

Note that the solution in terms of b_{m+4} must be compatible with Eq. (21). As a result, we get

$$b_{m+4}=\frac{kL^4}{60}f_{m+4}(a_0,a_1,a_2,\ldots,a_{m-1},a_m,m)=\frac{3kL^4a_m}{60(m+6)(m+7)}. \tag{31}$$

Thus, we arrive at an important restrictive condition signifying that the coefficients a_i cannot be chosen arbitrarily. Moreover, if any b_i coefficient is specified then the expression given in Eq. (30) is the final formula for the natural frequency.

To sum up, if the coefficients a_i satisfy Eq. (31), and any coefficient b_i is specified, both k and the other $m+4$ coefficients b_i could be ascertained by Eq. (30).

4.1. Example 1

Let $m=5$ and if $b_{m+4}=b$ is known, the determinant of the matrix $\mathbf{C}$, $|\mathbf{C}|=2047$. Thus $\mathbf{C}$ is reversible. From Eq. (22), we obtain

$$\mathbf{B}=\frac{kL^4}{60}\mathbf{C}^{-1}\mathbf{D}^{-1}\mathbf{E}\mathbf{A}, \tag{32}$$

$$\mathbf{c}^{-1} = \begin{bmatrix} 0.4998 & 0.2496 & 0.1246 & 0.0620 & 0.0308 & 0.0151 & 0.0073 & 0.0034 & 0.0015 & 0.0005 \\ 0.4993 & 0.7489 & 0.3737 & 0.1861 & 0.0923 & 0.0454 & 0.0220 & 0.0103 & 0.0044 & 0.0015 \\ 0.4983 & 0.7474 & 0.8720 & 0.4343 & 0.2154 & 0.1060 & 0.0513 & 0.0239 & 0.0103 & 0.0034 \\ 0.4963 & 0.7445 & 0.8686 & 0.9306 & 0.4617 & 0.2272 & 0.1099 & 0.0513 & 0.0220 & 0.0073 \\ 0.4924 & 0.7386 & 0.8617 & 0.9233 & 0.9541 & 0.4695 & 0.2272 & 0.1060 & 0.0454 & 0.0151 \\ 0.4846 & 0.7269 & 0.8481 & 0.9086 & 0.9389 & 0.9541 & 0.4617 & 0.2154 & 0.0923 & 0.0308 \\ 0.4690 & 0.7035 & 0.8207 & 0.8793 & 0.9086 & 0.9233 & 0.9306 & 0.4343 & 0.1861 & 0.0620 \\ 0.4377 & 0.6566 & 0.7660 & 0.8207 & 0.8481 & 0.8617 & 0.8686 & 0.8720 & 0.3737 & 0.1246 \\ 0.3752 & 0.5628 & 0.6566 & 0.7035 & 0.7269 & 0.7386 & 0.7445 & 0.7474 & 0.7489 & 0.2496 \\ 0.2501 & 0.3752 & 0.4377 & 0.4690 & 0.4846 & 0.4924 & 0.4963 & 0.4983 & 0.4993 & 0.4998 \end{bmatrix}. \quad (33)$$

We denote

$$\mathbf{C}^{-1}\mathbf{D}^{-1}\mathbf{E} = \mathbf{G}. \quad (34)$$

Hence, Eq. (32) becomes

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \\ b_8 \\ b_9 \end{bmatrix} = \frac{kL^4}{60} \begin{bmatrix} 0.0238 & 0.0047 & 0.0012 & 0.0003 & 0.0001 & 0.0000 \\ 0.0714 & 0.0142 & 0.0035 & 0.0010 & 0.0003 & 0.0001 \\ 0.0000 & 0.0332 & 0.0082 & 0.0023 & 0.0007 & 0.0002 \\ -0.1429 & -0.0121 & 0.0176 & 0.0050 & 0.0015 & 0.0005 \\ 0.0714 & -0.1029 & -0.0136 & 0.0103 & 0.0032 & 0.0011 \\ 0.0000 & 0.0490 & -0.0760 & -0.0124 & 0.0065 & 0.0022 \\ 0.0000 & -0.0044 & 0.0372 & -0.0578 & -0.0107 & 0.0044 \\ 0 & -0.0042 & -0.0042 & 0.0299 & -0.0451 & -0.0090 \\ 0.0000 & -0.0036 & -0.0036 & -0.0029 & 0.0250 & -0.0359 \\ 0.0000 & -0.0024 & -0.0024 & -0.0020 & -0.0015 & 0.0215 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}. \quad (35)$$

From Eq. (21), we get

$$b_9 = \frac{3kL^4 a_5}{60 \times 11 \times 12}. \quad (36)$$

Eq. (35) must be compatible with Eq. (36). Hence,

$$a_1 + a_2 + 0.833a_3 + 0.625a_4 + 0.5a_5 = 0. \quad (37)$$

The fundamental natural frequency squared reads

$$k = \frac{60 \times 11 \times 12}{3L^4 a_m} b_9 = \frac{2640}{a_m} b. \quad (38)$$

5. Specified flexural stiffness function

Consider now the case when the flexural stiffness function $D(\xi)$ of a longitudinally functionally graded beam is specified, implying that all b_i coefficients are given. The following question arises: Is it possible to determine the material density coefficients a_i?

There are $m+6$ Eqs. (12)–(21), while one has only $m+1$ unknowns, $a_0, \ldots, a_m$. In actuality, however, in order for the process of determining coefficients a_i to proceed, one of the coefficients a_i should be pre-specified. The most convenient assumption is to fix either a_0, a_1 or a_m, since in this case only one equation, in Eqs. (13), (14) or (21), respectively, will be sufficient to determine the sought expression of the natural frequency coefficient. Let us assume that the coefficient a_0 is given. Thus, we deduce from the compatibility of Eqs. (12)–(21), that five coefficients b_i, namely $b_0, b_{m+1}, b_{m+2}, b_{m+3}, b_{m+4}$, cannot be chosen arbitrarily.

Eliminating k from Eqs. (13)–(21), we get

$$a_1 = 2a_0 \frac{2b_1 - 3b_2 + b_3}{2b_0 - 3b_1 + b_2}, \tag{39}$$

$$a_2 = 10a_0 + \frac{10}{3} a_0 \frac{2b_2 - 3b_3 + b_4}{2b_0 - 3b_1 + b_2}, \tag{40}$$

$$a_3 = -15a_0 + 10a_1 + 5a_0 \frac{2b_3 - 3b_4 + b_5}{2b_0 - 3b_1 + b_2}, \tag{41}$$

$$a_{i-1} = 6a_{i-5} - 15a_{i-4} + 10a_{i-3} + \frac{(i+1)(i+2)}{6} a_0 \frac{2b_{i-1} - 3b_i + b_{i+1}}{2b_0 - 3b_1 + b_2}, \quad (5 \leqslant i \leqslant m+1). \tag{42}$$

The coefficients b_i must satisfy the following conditions:

(1) All coefficients b_i cannot be equal to each other, otherwise a_i cannot be determined.
(2) In Eq. (13), in order for the physical realizability condition $k = \omega^2 > 0$ to be satisfied, the expression $2b_0 - 3b_1 + b_2$ and the coefficient a_0 should have opposite signs. Also, since $D(0) > 0$, the coefficient b_0 must be positive.
(3) The coefficient $b_0, b_{m+1}, b_{m+2}, b_{m+3}, b_{m+4}$, must be related as follows:

$$b_0 = \frac{1}{3} b_1, \quad b_{m+4} = \frac{-18a_m(2b_0 - 3b_1 + b_2)}{a_0(m+6)(m+7)}, \tag{43,44}$$

$$b_{m+3} = \left[\frac{m+7}{m+5} \cdot \frac{6a_{m-1} - 15a_m}{6a_m} + \frac{3}{2}\right] b_{m+4}, \tag{45}$$

$$b_{m+2} = \left[\frac{m+6}{m+4} \cdot \frac{6a_{m-2} - 15a_{m-1} + 10a_m}{6a_{m-1} - 15a_m} + \frac{3}{2}\right] b_{m+3} - \left[\frac{m+6}{m+4} \cdot \frac{6a_{m-2} - 15a_{m-1} + 10a_m}{6a_{m-1} - 15a_m} \cdot \frac{3}{2} + \frac{1}{2}\right] b_{m+4}, \tag{46}$$

$$b_{m+1} = \left(\alpha + \frac{3}{2}\right) b_{m+2} - \left(\alpha \cdot \frac{3}{2} + \frac{1}{2}\right) b_{m+3} + \alpha \cdot \frac{1}{2}\, b_{m+4}, \tag{47}$$

where

$$\alpha = \frac{m+5}{m+3} \cdot \frac{6a_{m-3} - 15a_{m-2} + 10a_{m-1} - a_{m+1}}{6a_{m-2} - 15a_{m-1} + 10a_m}. \tag{48}$$

To sum up, when the coefficients b_i $(b_0, \ldots, b_m)$ and a_0 satisfy the conditions as specified above, we can obtain the remaining coefficients a_i from Eqs. (39)–(42) and the remaining five coefficients b_i from Eqs. (42)–(48). The fundamental natural frequently squared reads

$$\omega^2 = k = \frac{360(-2b_0 + 3b_1 - b_2)}{a_0 L^4}. \tag{49}$$

6. Particular case: $m<4$

Let us discuss the particular case $m<4$. The inertial coefficients are specified as follows.

6.1. Sub-case a: $m = 0,\ R(\xi) = a_0,\ b_4 = b$

Eq. (11) results in

$$\begin{aligned} &-60\sum_{i=1}^{3} i(i+1)b_{i+1}\xi^i + 180\sum_{i=2}^{4} i(i-1)b_i\xi^i - 120\sum_{i=3}^{5}(i-1)(i-2)b_{i-1}\xi^i + 360\sum_{i=0}^{4} b_i\xi^i \\ &\quad - 720\sum_{i=1}^{5} b_{i-1}\xi^i - 120\sum_{i=0}^{3}(i+1)b_{i+1}\xi^i + 720\sum_{i=1}^{4} ib_i\xi^i - 720\sum_{i=2}^{5}(i-1)b_{i-1}\xi^i \\ &\quad - kL^4\sum_{i=1}^{1} a_{i-1}\xi^i + 10kL^4\sum_{i=3}^{3} a_{i-3}\xi^i - 15kL^4\sum_{i=4}^{4} a_{i-4}\xi^i + 6kL^4\sum_{i=5}^{5} a_{i-5}\xi^i = 0. \end{aligned} \tag{50}$$

Since Eq. (50) has to be satisfied for any ξ, the following relations are obtained:

$$\begin{gathered} 360b_0 - 120b_1 = 0, \quad -720b_0 + 1080b_1 - 360b_2 - kL^4a_0 = 0, \\ -1440b_1 + 2160b_2 - 720b_3 = 0, \quad -240b_2 + 360b_3 - 120b_4 + kL^4a_0 = 0, \\ -240b_3 + 360b_4 - kL^4a_0 = 0, \quad -840b_4 - kL^4a_0 = 0. \end{gathered} \tag{51}$$

This set is compatible and has a unique solution. The coefficient a_0 can be chosen arbitrarily. Then,

$$b_0 = \frac{1}{3}\, b_4 = \frac{1}{3}\, b, \quad b_1 = b, \quad b_2 = 0, \quad b_3 = -2b, \quad k = \frac{840b}{a_0L^4}. \tag{52}$$

6.2. Sub-case b: $m = 1,\ R(\xi) = a_0 + a_1\xi,\ b_5 = b$

Eq. (11) can be re-written as

$$-60\sum_{i=1}^{4} i(i+1)b_{i+1}\xi^i + 180\sum_{i=2}^{5} i(i-1)b_i\xi^i - 120\sum_{i=3}^{6}(i-1)(i-2)b_{i-1}\xi^i + 360\sum_{i=0}^{5} b_i\xi^i$$

$$-720\sum_{i=1}^{6} b_{i-1}\xi^i - 120\sum_{i=0}^{4}(i+1)b_{i+1}\xi^i + 720\sum_{i=1}^{5} ib_i\xi^i - 720\sum_{i=2}^{6}(i-1)b_{i-1}\xi^i$$
$$-kL^4\sum_{i=1}^{2} a_{i-1}\xi^i + 10kL^4\sum_{i=3}^{4} a_{i-3}\xi^i - 15kL^4\sum_{i=4}^{5} a_{i-4}\xi^i + 6kL^4\sum_{i=5}^{6} a_{i-5}\xi^i = 0. \quad (53)$$

It has to be satisfied for any ξ. This requirement yields the following relations:

$$360b_0 - 120b_1 = 0,$$

$$-720b_0 + 1080b_1 - 360b_2 - kL^4a_0 = 0,$$

$$-1440b_1 + 2160b_2 - 720b_3 - kL^4a_1 = 0,$$

$$-240b_2 + 360b_3 - 120b_4 + kL^4a_0 = 0,$$

$$-720b_3 + 1080b_4 - 360b_5 + 2kL^4a_1 - 3kL^4a_0 = 0,$$

$$-1680b_4 + 2520b_5 - 5kL^4a_1 + 2kL^4a_0 = 0, \quad 1120b_5 = kL^4a_1. \quad (54)$$

The solution of this set reads

$$b_4 = \frac{8a_0 - 11a_1}{6a_1}\,b, \quad b_3 = \frac{-96a_0 - 5a_1}{36a_1}\,b, \quad b_2 = \frac{17}{24}\,b,$$
$$b_1 = \frac{64a_0 + 17a_1}{48a_1}b, \quad b_0 = \frac{64a_0 + 17a_1}{144a_1}b, \quad k = \frac{1120b}{L^4a_1}. \quad (55)$$

In order for the set to be compatible, a_1 must vanish. Therefore, we conclude that the set has no solution.

6.3. Sub-case c: $m = 2,\ R(\xi) = a_0 + a_1\xi + a_2\xi^2,\ b_6 = b$

Eq. (11) can be re-written as

$$-60\sum_{i=1}^{5} i(i+1)b_{i+1}\xi^i + 180\sum_{i=2}^{6} i(i-1)b_i\xi^i - 120\sum_{i=3}^{7}(i-1)(i-2)b_{i-1}\xi^i + 360\sum_{i=0}^{6} b_i\xi^i$$
$$-720\sum_{i=1}^{7} b_{i-1}\xi^i - 120\sum_{i=0}^{5}(i+1)b_{i+1}\xi^i + 720\sum_{i=1}^{6} ib_i\xi^i - 720\sum_{i=2}^{7}(i-1)b_{i-1}\xi^i$$
$$-kL^4\sum_{i=1}^{3} a_{i-1}\xi^i + 10kL^4\sum_{i=3}^{5} a_{i-3}\xi^i - 15kL^4\sum_{i=4}^{6} a_{i-4}\xi^i + 6kL^4\sum_{i=5}^{7} a_{i-5}\xi^i = 0. \quad (56)$$

It has to be satisfied for any ξ. This requirement yields the following relations:

$$360b_0 - 120b_1 = 0,$$

$$-720b_0 + 1080b_1 - 360b_2 - kL^4a_0 = 0,$$

$$-1440b_1 + 2160b_2 - 720b_3 - kL^4a_1 = 0,$$

$$-2400b_2 + 3600b_3 - 1200b_4 - kL^4a_2 + 10kL^4a_0 = 0,$$

$$-720b_3 + 1080b_4 - 360b_5 + 2kL^4a_1 - 3kL^4a_0 = 0,$$

$$-5040b_4 + 7560b_5 - 2520b_6 + 10kL^4a_2 - 15kL^4a_1 + 6kL^4a_0 = 0,$$

$$-2240b_5 + 3360b_6 - 5kL^4a_2 + 2kL^4a_1 = 0,$$

$$-1440b_6 + kL^4a_2 = 0. \quad (57)$$

The solution of this set reads

$$b_5 = \frac{-12a_2 + 9a_1}{7a_2}\,b, \quad b_4 = \frac{-3a_2 - 33a_1 + 24a_0}{14a_2}\,b, \quad b_3 = \frac{15a_2 - 5a_1 - 96a_0}{28a_2}\,b,$$
$$b_2 = \frac{87a_2 + 255a_1}{280a_2}\,b, \quad b_1 = \frac{111a_2 + 255a_1 + 960a_0}{560a_2}\,b,$$
$$b_0 = \frac{318a_2 + 51a_1 + 128a_0}{224a_2}\,b, \quad k = \frac{1440b}{L^4a_2}. \quad (58)$$

In order for the set to be compatible, we have to improve the following requirement:

$$a_2 = a_1. \quad (59)$$

6.4. Sub-case d: $m = 3$, $R(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$, $b_7 = b$

Eq. (11) can be re-written as

$$\begin{aligned}
&-60\sum_{i=1}^{6} i(i+1)b_{i+1}\xi^i + 180\sum_{i=2}^{7} i(i-1)b_i\xi^i - 120\sum_{i=3}^{8}(i-1)(i-2)b_{i-1}\xi^i + 360\sum_{i=0}^{7} b_i\xi^i \\
&\quad - 720\sum_{i=1}^{8} b_{i-1}\xi^i - 120\sum_{i=0}^{6}(i+1)b_{i+1}\xi^i + 720\sum_{i=1}^{7} ib_i\xi^i - 720\sum_{i=2}^{8}(i-1)b_{i-1}\xi^i \\
&\quad - kL^4\sum_{i=1}^{4} a_{i-1}\xi^i + 10kL^4\sum_{i=3}^{6} a_{i-3}\xi^i - 15kL^4\sum_{i=4}^{7} a_{i-4}\xi^i + 6kL^4\sum_{i=5}^{8} a_{i-5}\xi^i = 0.
\end{aligned} \quad (60)$$

It has to be satisfied for any ξ. This requirement yields the following relations:

$$360b_0 - 120b_1 = 0,$$

$$-720b_0 + 1080b_1 - 360b_2 - kL^4a_0 = 0,$$

$$-1440b_1 + 2160b_2 - 720b_3 - kL^4a_1 = 0,$$

$$-2400b_2 + 3600b_3 - 1200b_4 - kL^4a_2 + 10kL^4a_0 = 0,$$

$$
\begin{gathered}
-3600b_3 + 5400b_4 - 1800b_5 - kL^4a_3 10kL^4a_1 - 15kL^4a_0 = 0, \\
-5040b_4 + 7560b_5 - 2520b_6 + 10kL^4a_2 - 15kL^4a_1 + 6kL^4a_0 = 0, \\
-6720b_5 + 10\,080b_6 - 3360b_7 + 10kL^4a_3 - 15kL^4a_2 + 6kL^4a_1 = 0, \\
-2880b_6 + 4320b_7 - 5kL^4a_3 + 2kL^4a_2 = 0, \\
-1800b_7 + kL^4a_3 = 0. \qquad (61)
\end{gathered}
$$

The solution of this set is given by

$$
\begin{aligned}
& b_6 = \frac{-13a_3 + 10a_2}{8a_3}b, \quad b_5 = \frac{-29a_3 - 240a_2 + 180a_1}{112a_3}b, \\
& b_4 = \frac{95a_3 - 60a_2 - 660a_1 + 480a_0}{224a_3}b, \quad b_3 = \frac{119a_3 + 300a_3 - 100a_1 - 1920a_0}{448a_3}b, \\
& b_2 = \frac{167a_3 + 348a_2 + 1020a_1}{896a_3}b, \quad b_1 = \frac{263a_3 + 444a_2 + 1020a_1 + 3840a_0}{1792a_3}b, \\
& b_0 = \frac{455a_3 + 636a_2 + 1020a_1 + 2560a_0}{3584a_3}b, \quad k = \frac{1800b}{L^4a_3}. \qquad (62)
\end{aligned}
$$

In order for the set to be compatible, the following condition must be met:

$$
a_3 + 1.22a_2 + 1.22a_1 = 0. \qquad (63)
$$

7. Conclusion

The reported results indicate that the proposed method is feasible when the inertial coefficient, flexural stiffness and the mode shape all are polynomial functions. It is hoped that when the axially graded materials will be developed this study, and the attendant ones, will gain considerable practical interest.

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